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# STUDY OF THE TEMPERATURE FIELDS IN OIL AND GAS STRATA WITH WATER INJECTION BASED ON THE PERTURBATION METHOD

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*The problem of the thermal action on oil and gas strata by injection of a heating medium using expansion in a small parameter is reduced to an infinite sequence of boundary-value problems that are solved by the method of integral transforms. It is shown that, with an appropriate selection of the small parameter, the zeroth-order approximation corresponds to a space-averaged (across the stratum thickness) solution of the main problem and leads to a "concentrated-capacity scheme" that is constructed assuming that the stratum temperature is independent of the vertical coordinate. The first approximation permitted marked refinement of calculations according to the "concentrated-capacity scheme" and an evaluation of its error. Space-time temperature distributions are presented that have been calculated using the analytical solutions obtained.*

**Introduction.** The pressure in oil and gas strata is usually maintained by injecting water. A detrimental side effect here is stratum cooling, which reduces the oil output. The temperature fields in oil and gas strata are studied in a great many works of the scientific schools of Kazan and Lithuanian State Universities, scientific-research and design institutes of the oil and gas industry [1-4], and foreign researchers [5]. The oil output is increased using injection of a heating medium. Interest in the problem of the temperature fields in oil and gas strata is also fairly great in connection with various geophysical applications.

Calculation of the temperature fields with water injection into oil and gas strata necessitates solution of problems of convective heat conduction. The complete system of equations describing these problems is rather complicated and contains the continuity equation, the equation of motion, the energy equation, and the equation of state of the substance. Solutions of this system are difficult to obtain in general form; therefore simplifications should be introduced.

Among the most efficient methods of simplification of such problems is the "concentrated-capacity" method, which assumes that the stratum temperature is invariable across the thickness [1-4]. However, the concentrated-capacity scheme has drawbacks of its own. It is suitable only for calculating mean values and offers no opportunity for evaluating the error arising in the calculations.

Below, a method is proposed for simplifying the problem of convective heat conduction using a small parameter. When it is selected appropriately, the zeroth-order approximation coincides with the concentrated-capacity scheme, and higher approximations permit an evaluation of the error introduced in the calculation.

In the work proposed, consideration is given to a two-dimensional axisymmetric problem in a cylindrical coordinate system that describes the temperature distribution with water injection into porous oil and gas strata.

**1. Problem Formulation.** We assume that water with a specified temperature is injected into a horizontal stratum of thickness $-h < z_d < h$ through a small-radius well (Fig. 1). For simplicity, the temperatures of the skeleton of the porous medium and the incompressible fluid that saturates it are supposed to be identical,

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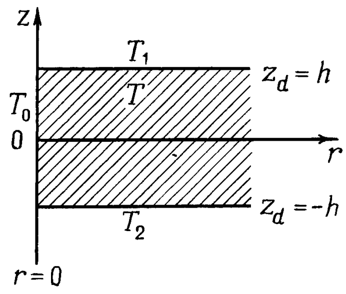

Fig. 1. Geometry of the problem.

since the heat transfer between the skeleton and the fluid proceeds fairly rapidly. This assumption is fulfilled, since the specific surface of porous oil and gas strata is large. The fluid is regarded as incompressible, for simplicity phase changes are disregarded, and capillary forces, gravity, and temperature variations of the volumes and the thermal properties of the system in question are neglected.

A constant temperature $T_0$ is maintained in the fluid arriving at the stratum at $r = 0$. When hot fluid is injected, heat propagation in the stratum occurs by convection along the radial axis and by heat conduction in the vertical direction. Concurrently, heat transfer occurs by heat conduction between the stratum and the adjoining, overlying and underlying, rocks; radial heat conduction is neglected. As a result of the joint action of the indicated factors, the stratum temperature is a function of two space coordinates and time. Conditions of equality of the temperatures and the heat fluxes at the stratum boundaries are postulated, and initial and boundary conditions are imposed.

The mathematical formulation of the problem in dimensionless coordinates is of the form

$$\frac{\partial T_1}{\partial t} = \frac{\partial^2 T_1}{\partial z^2}, \quad t > 0, \quad z > 1; \tag{1.1}$$

$$\frac{\lambda_1 b_2}{\lambda_2} \frac{\partial T_2}{\partial t} = \frac{\partial^2 T_2}{\partial z^2}, \quad t > 0, \quad z < -1; \tag{1.2}$$

$$\varepsilon b \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} - \varepsilon \frac{B}{\lambda_1 r} \frac{\partial T}{\partial r}, \quad t > 0, \quad r > 0, \quad |z| < 1; \tag{1.3}$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=1} = \varepsilon \left.\frac{\partial T_1}{\partial z}\right|_{z=1}; \quad \left.\frac{\partial T}{\partial z}\right|_{z=-1} = \varepsilon \frac{\lambda_2}{\lambda_1} \left.\frac{\partial T_2}{\partial z}\right|_{z=-1}; \tag{1.4}$$

$$T|_{z=1} = T_1|_{z=1}; \quad T|_{z=-1} = T_2|_{z=-1}; \tag{1.5}$$

$$T|_{t=0} = 0; \quad T_1|_{t=0} = 0; \quad T_2|_{t=0} = 0; \tag{1.6}$$

$$T|_{r=0} = 1; \quad T|_{r\to+\infty} = 0; \quad T_1|_{z\to+\infty} = 0; \quad T_2|_{z\to-\infty} = 0, \tag{1.7}$$

where

$$t = \frac{\tau\lambda_1}{\rho_1 c_1 h^2}, \quad z = \frac{z_d}{h}, \quad r = \frac{r_d}{h}, \quad B = \frac{Q\rho c}{4\pi h}, \quad b = \frac{\rho c}{\rho_1 c_1}, \quad b_2 = \frac{\rho_2 c_2}{\rho_1 c_1}, \quad T = \frac{T_d}{T_0}, T_1 = \frac{T_{d1}}{T_0}, \quad T_2 = \frac{T_{d2}}{T_0}, \quad \varepsilon = \frac{\lambda_1}{\lambda}.$$

**2. Expansion of Solutions in Power Series.** The coefficient ε entering into Eqs. (1.3) and (1.4) is in most cases smaller than unity. This can be explained proceeding from the following reasoning. As a rule, there is no convective heat transfer in the rocks surrounding the stratum; therefore the thermal conductivity is represented by the molecular component. In the stratum with the water injection, the convective component is prevalent; the thermal conductivity is the sum of the coefficients of molecular and convective heat conduction (the ratio $\lambda_1/\lambda << 1$).

From the foregoing it also follows that a small parameter in the problem in question can be introduced formally and afterward set equal to unity, since the radius of convergence of the obtained series increase indefinitely with time.

With the small parameter ε, it is expedient to seek the solution in the form of series of perturbation theory:

$$T = T^{(0)} + \varepsilon T^{(1)} + \varepsilon^2 T^{(2)} + \ldots , \quad T_1 = T_1^{(0)} + \varepsilon T_1^{(1)} + \varepsilon^2 T_1^{(2)} + \ldots ,$$

$$T_2 = T_2^{(0)} + \varepsilon T_2^{(1)} + \varepsilon^2 T_2^{(2)} + \ldots . \tag{2.1}$$

The superscript in parentheses here and subsequently corresponds to the order of expansion in ε.

Certain ideas of the application of the method of a small parameter to heat-conduction problems are described in [6].

In this work, selection of the small parameter for the system of equations is proposed such that the zeroth-order approximation results in a concentrated-capacity scheme [1-4]. Furthermore, solutions of the problem in the zeroth-order and first approximations are obtained, and the method of finding higher approximations is described.

**3. Simplification of the System of Equations by the Small Parameter.** Having substituted expression (2.1) into relations (1.1)-(1.7) and having grouped terms of the same order in ε, we obtain the following system of equations:

$$\frac{\partial T_1^{(i)}}{\partial t} = \frac{\partial^2 T_1^{(i)}}{\partial z^2} , \quad i = 0, 1, 2, \ldots ; \tag{3.1}$$

$$\frac{\lambda_1 b_2}{\lambda_2} \frac{\partial T_2^{(i)}}{\partial t} = \frac{\partial^2 T_2^{(i)}}{\partial z^2} , \quad i = 0, 1, 2, \ldots ; \tag{3.2}$$

$$\frac{\partial^2 T^{(0)}}{\partial z^2} = 0 ; \tag{3.3}$$

$$b \frac{\partial T^{(i-1)}}{\partial t} = \frac{\partial^2 T^{(i)}}{\partial z^2} - \frac{B}{\lambda_1 r} \frac{\partial T^{(i-1)}}{\partial r} , \quad i = 1, 2, 3, \ldots ; \tag{3.4}$$

$$\left. \frac{\partial T^{(0)}}{\partial z} \right|_{z=1} = 0 ; \quad \left. \frac{\partial T^{(0)}}{\partial z} \right|_{z=-1} = 0 ; \tag{3.5}$$

$$\left. \frac{\partial T^{(i)}}{\partial z} \right|_{z=1} = \left. \frac{\partial T_1^{(i-1)}}{\partial z} \right|_{z=1} ; \quad \left. \frac{\partial T^{(i)}}{\partial z} \right|_{z=-1} = \frac{\lambda_2}{\lambda_1} \left. \frac{\partial T_2^{(i-1)}}{\partial z} \right|_{z=-1} ,$$

$$i = 1, 2, 3, \dots ; \tag{3.6}$$

$$T^{(i)}|_{z=1} = T_1^{(i)}|_{z=1} ; \quad T^{(i)}|_{z=-1} = T_2^{(i)}|_{z=-1} , \quad i = 0, 1, 2, \dots ; \tag{3.7}$$

$$T^{(i)}|_{t=0} = 0 ; \quad T_1^{(i)}|_{t=0} = 0 ; \quad T_2^{(i)}|_{t=0} = 0 , \quad i = 0, 1, 2, \dots ; \tag{3.8}$$

$$T^{(0)}|_{r=0} = 1 ; \tag{3.9}$$

$$T^{(i)}|_{r\to+\infty} = 0 ; \quad T_1^{(i)}|_{z\to+\infty} = 0 ; \quad T_2^{(i)}|_{z\to-\infty} = 0 , \quad i = 0, 1, 2, \dots . \tag{3.10}$$

The systems of equations (3.1)-(3.10) with a specified $i$ permit a problem formulation for the approximation of the corresponding order.

**4. Solution of the Problem in the Zeroth-Order Approximation.** From expression (3.4) for $i = 1$ we obtain

$$b \frac{\partial T^{(0)}}{\partial t} = \frac{\partial^2 T^{(1)}}{\partial z^2} - \frac{B}{\lambda_1 r} \frac{\partial T^{(0)}}{\partial r} . \tag{4.1}$$

According to Eqs. (3.3) and (3.5), the solution of the problem for the zeroth-order approximation is independent of $z$ and has the form $T^{(0)} = A(t, r)$, where $A(t, r)$ is an indefinite function of the time $t$ and the coordinate $r$. Then, all terms of Eq. (4.1) that contain the temperature in the zeroth-order approximation are independent of $z$. Let us write

$$b \frac{\partial T^{(0)}}{\partial t} + \frac{B}{\lambda_1 r} \frac{\partial T^{(0)}}{\partial r} \equiv R\,(t, r) , \tag{4.2}$$

where $R(t, r)$ is a function that is independent of $z$.

With account for expression (4.2), Eq. (4.1) is written as

$$\frac{\partial^2 T^{(1)}}{\partial z^2} = R\,(t, r) . \tag{4.3}$$

Having integrated Eq. (4.3) twice, we obtain an expression for the stratum temperature in the first approximation:

$$\frac{\partial T^{(1)}}{\partial z} = zR\,(t, r) + N\,(t, r) , \tag{4.4}$$

$$T^{(1)} = \frac{z^2}{2} R\,(t, r) + zN\,(t, r) + G\,(t, r) . \tag{4.5}$$

From Eq. (3.6) for $i = 1$ and Eq. (4.4) at the boundaries $z = 1$ and $z = -1$, we obtain a system of two equations for $R(t, r)$ and $N(t, r)$, whence

$$R\,(t, r) = \frac{1}{2} \frac{\partial T_1^{(0)}}{\partial z}\bigg|_{z=1} - \frac{1}{2} \frac{\lambda_2}{\lambda_1} \frac{\partial T_2^{(0)}}{\partial z}\bigg|_{z=-1} , \tag{4.6}$$

$$N(t, r) = \frac{1}{2} \frac{\partial T_1^{(0)}}{\partial z}\bigg|_{z=1} + \frac{1}{2} \frac{\lambda_2}{\lambda_1} \frac{\partial T_2^{(0)}}{\partial z}\bigg|_{z=-1} . \quad (4.7)$$

Assuming $i = 0$ in Eqs. (3.1), (3.2), (3.7), (3.8), and (3.10) and taking account of Eqs. (4.1), (4.2), and (4.6), we obtain the following boundary-value problem in the zeroth-order approximation:

$$\frac{\partial T_1^{(0)}}{\partial t} = \frac{\partial^2 T_1^{(0)}}{\partial z^2} , \quad t > 0 , \quad z > 1 ; \quad (4.8)$$

$$\frac{\lambda_1 b_2}{\lambda_2} \frac{\partial T_2^{(0)}}{\partial t} = \frac{\partial^2 T_2^{(0)}}{\partial z^2} , \quad t > 0 , \quad z < -1 ; \quad (4.9)$$

$$b \frac{\partial T^{(0)}}{\partial t} = -\frac{B}{\lambda_1 r} \frac{\partial T^{(0)}}{\partial r} + \frac{1}{2} \frac{\partial T_1^{(0)}}{\partial z}\bigg|_{z=1} -$$

$$- \frac{1}{2} \frac{\lambda_2}{\lambda_1} \frac{\partial T_2^{(0)}}{\partial z}\bigg|_{z=-1} , \quad t > 0 , \quad r > 0 , \quad |z| < 1 ; \quad (4.10)$$

$$T^{(0)} = T_1^{(0)}|_{z=1} = T_2^{(0)}|_{z=-1} ; \quad (4.11)$$

$$T^{(0)}|_{t=0} = 0 ; \quad T_1^{(0)}|_{t=0} = 0 ; \quad T_2^{(0)}|_{t=0} = 0 ; \quad (4.12)$$

$$T^{(0)}|_{r=0} = 1 ; \quad T^{(0)}|_{r\to+\infty} = 0 ; \quad T_1^{(0)}|_{z\to+\infty} = 0 ; \quad T_2^{(0)}|_{z\to-\infty} = 0 . \quad (4.13)$$

Condition (4.11) follows from Eq. (3.7) for $i = 0$ and the aforementioned $z$-independence of $T^{(0)}$.

The obtained boundary-value problem (4.8)-(4.13) coincides with the problem obtained according to the concentrated-capacity scheme [1-4]. It is solved using the Laplace–Carson transform. In the image space we obtain

$$T_1^{(0)P} = T^{(0)P} \exp\left(-\sqrt{p}\,(z-1)\right) , \quad (4.14)$$

$$T_2^{(0)P} = T^{(0)P} \exp\left(\sqrt{\left(\frac{\lambda_1 b_2}{\lambda_2}\right)} p\,(z+1)\right), \quad (4.15)$$

$$T^{(0)P} = \exp\left(-\frac{\lambda_1 r^2}{2B}\left[\frac{1}{2}\left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)\sqrt{p} + bp\right]\right). \quad (4.16)$$

The superscript $P$ here and subsequently indicates that the corresponding expressions are written in the image space.

Converting to inverse transforms, we obtain expressions for the temperature in the stratum and the surrounding rocks:

$$T^{(0)} = \operatorname{erfc}\left(\frac{\lambda_1 r^2\left(1+\sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)}{8B\sqrt{\left(t-\frac{\lambda_1 br^2}{2B}\right)}}\right) I\left(t-\frac{\lambda_1 br^2}{2B}\right), \tag{4.17}$$

$$T_1^{(0)} = \operatorname{erfc}\left(\frac{\frac{\lambda_1 r^2}{4B}\left(1+\sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)+(z-1)}{2\sqrt{\left(t-\frac{\lambda_1 br^2}{2B}\right)}}\right) I\left(t-\frac{\lambda_1 br^2}{2B}\right), \tag{4.18}$$

$$T_2^{(0)} = \operatorname{erfc}\left(\frac{\frac{\lambda_1 r^2}{4B}\left(1+\sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)-\sqrt{\left(\frac{\lambda_1 b_2}{\lambda_2}\right)}(z+1)}{2\sqrt{\left(t-\frac{\lambda_1 br^2}{2B}\right)}}\right) I\left(t-\frac{\lambda_1 br^2}{2B}\right). \tag{4.19}$$

The zeroth-order asymptotic approximations (4.17)-(4.19) for the stratum and the surrounding rocks coincide with solutions obtained according to the concentrated-capacity scheme and conform to the actual temperature distribution. It is known that the concentrated-capacity scheme is suitable for calculation of the mean temperatures. However, it does not permit an evaluation of the error that arises; for this, subsequent approximations must be considered. It should be noted that finding the first, second, etc. approximations requires additional conditions.

**5. Problem Formulation in the First Approximation.** Let us find an expression for the stratum temperature in the first approximation. For $i = 2$, Eq. (3.4) takes the form

$$b\frac{\partial T^{(1)}}{\partial t} = \frac{\partial^2 T^{(2)}}{\partial z^2} - \frac{B}{\lambda_1 r}\frac{\partial T^{(1)}}{\partial r}. \tag{5.1}$$

From Eqs. (5.1) and (4.5) we obtain

$$\frac{\partial^2 T^{(2)}}{\partial z^2} = \frac{z^2}{2}\hat{L}R(t, r) + z\hat{L}N(t, r) + \hat{L}G(t, r), \tag{5.2}$$

where

$$\hat{L} = b\frac{\partial}{\partial t} + \frac{B}{\lambda_1 r}\frac{\partial}{\partial r}.$$

From Eq. (3.6) for $i = 2$ and on integrating Eq. (5.2) at the boundaries $z = 1$ and $z = -1$, we obtain a system of two equations for $\hat{L}G(t, r)$ and $M(t, r)$:

$$\hat{L}G(t, r) = \frac{1}{2}\frac{\partial T_1^{(1)}}{\partial z}\bigg|_{z=1} - \frac{1}{2}\frac{\lambda_2}{\lambda_1}\frac{\partial T_2^{(1)}}{\partial z}\bigg|_{z=-1} - \frac{1}{6}\hat{L}R(t, r), \tag{5.3}$$

$$M(t,r) = \frac{1}{2}\frac{\partial T_1^{(1)}}{\partial z}\bigg|_{z=1} + \frac{1}{2}\frac{\lambda_2}{\lambda_1}\frac{\partial T_2^{(1)}}{\partial z}\bigg|_{z=-1} - \frac{1}{2}\hat{L}N(t,r)\,. \tag{5.4}$$

Having substituted the expressions for $R(t, r)$ and $N(t, r)$ from Eqs. (4.6) and (4.7) and for $\hat{L}G(t, r)$ from Eq. (5.3) into Eq. (5.2), we formulate, based on Eq. (5.1), a mathematical statement of the problem for the first expansion coefficients:

$$\frac{\partial T_1^{(1)}}{\partial t} = \frac{\partial^2 T_1^{(1)}}{\partial z^2}\,, \quad t > 0\,, \quad z > 1\,; \tag{5.5}$$

$$\frac{\lambda_1 b_2}{\lambda_2}\frac{\partial T_2^{(1)}}{\partial t} = \frac{\partial^2 T_2^{(1)}}{\partial z^2}\,, \quad t > 0\,, \quad z < -1\,; \tag{5.6}$$

$$b\frac{\partial T^{(1)}}{\partial t} + \frac{B}{\lambda_1 r}\frac{\partial T^{(1)}}{\partial r} - \frac{1}{2}\frac{\partial T_1^{(1)}}{\partial z}\bigg|_{z=1} + \frac{1}{2}\frac{\lambda_2}{\lambda_1}\frac{\partial T_2^{(1)}}{\partial z}\bigg|_{z=-1} =$$

$$= \left(\frac{z^2}{4} + \frac{z}{2} - \frac{1}{12}\right)\hat{L}\left(\frac{\partial T_1^{(0)}}{\partial z}\bigg|_{z=1}\right) -$$

$$-\left(\frac{z^2}{4} - \frac{z}{2} - \frac{1}{12}\right)\hat{L}\left(\frac{\lambda_2}{\lambda_1}\frac{\partial T_2^{(0)}}{\partial z}\bigg|_{z=-1}\right), \quad t > 0\,, \quad r > 0\,, \quad |z| < 1\,; \tag{5.7}$$

$$T^{(1)}|_{z=1} = T_1^{(1)}|_{z=1}\,; \quad T^{(1)}|_{z=-1} = T_2^{(1)}|_{z=-1}\,; \tag{5.8}$$

$$T^{(1)}|_{t=0} = 0\,; \quad T_1^{(1)}|_{t=0} = 0\,; \quad T_2^{(1)}|_{t=0} = 0\,; \tag{5.9}$$

$$T^{(1)}|_{r\to+\infty} = 0\,; \quad T_1^{(1)}|_{r\to+\infty} = 0\,; \quad T_2^{(1)}|_{r\to-\infty} = 0\,. \tag{5.10}$$

However, solution of problem (5.5)-(5.10) necessitates an additional condition that is derived below.

**6. Obtaining Additional Conditions for Determination of the Temperature in the First and Higher Approximations.** In order to find an additional condition for determining $T^{(1)}$, Eq. (1.3) is averaged with respect to $z$ between the limits –1 and 1 using determination of the mean value and equalities (1.4).

It is not difficult to assure ourselves that the boundary-value problem for determining the mean temperature in the stratum and the corresponding temperatures in the surrounding rocks coincides with the boundary-value problem for the zeroth-order approximation (4.8)-(4.13). From the uniqueness of the solution of the corresponding problems for the mean values $\langle T\rangle$ and the zeroth-order approximation $T^{(0)}$ it follows that $\langle T\rangle = T^{(0)}$. On the other hand, upon averaging, from expression (2.1) we obtain $\langle T\rangle = T^{(0)} + \varepsilon\langle T^{(1)}\rangle + \varepsilon^2\langle T^{(2)}\rangle + \dots$ . Thus, for all expansion coefficients of order above zeroth the following equalities must be fulfilled:

$$\langle T^{(1)}\rangle = 0\,, \quad \langle T^{(2)}\rangle = 0\,, \quad \dots\,, \quad \langle T^{(i)}\rangle = 0\,, \quad \dots\,. \tag{6.1}$$

It is the equality $\langle T^{(1)}\rangle = 0$ that is the sought additional condition for determining the first expansion coefficients. Averaging expression (4.5) over the surface $r = r_0$, we obtain an expression relating $R(t, r)$ to $G(t, r)$:

$$G\,(t, r_0) = -\frac{1}{6} R\,(t, r_0)\,. \tag{6.2}$$

From the foregoing it follows that the asymptotic solution conforms to the solution of the initial problem averaged over the vertical coordinate in the interval of the stratum. We also showed that the expansion coefficients thus obtained correspond to the Taylor coefficients of expansion of the exact solution of the initial problem only if it is additionally required that condition (6.2) be fulfilled on the line where the boundary condition is specified, i.e., for $r_0 = 0$.

**7. Solution of the Problem in the First Approximation.** Solution of Eqs. (5.5) and (5.6) offers the opportunity to obtain expressions for the first expansion coefficients in the image space for the surrounding rocks:

$$T_1^{(1)P} = T^{(1)P}\big|_{z=1} \exp\,(-\sqrt{p}\,(z-1))\,, \tag{7.1}$$

$$T_2^{(1)P} = T^{(1)P}\big|_{z=-1} \exp\left(\sqrt{\left(\frac{\lambda_1 b_2}{\lambda_2}\right)} p\,(z+1)\right). \tag{7.2}$$

Equation (5.7) for the first expansion coefficient in the stratum is equivalent to Eq. (5.3) for the function $G(t, r)$. The integration constant for the solution of Eq. (5.3) is found from condition (6.2).

Expression (4.5) with account for Eqs. (4.6) and (4.7) and the expression for $G(t, r)$, which in the image space is written as

$$G^P\,(p, r) = \left(\frac{\sqrt{p}}{12}\left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right) + \frac{\lambda_1 b\,(r^2 - r_0^2)\,p}{8B}\left[\frac{1}{3}\times\right.\right.$$

$$\left.\left.\times\left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)^2 + \left(1 - \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)^2\right]\right) T^{(0)P}\,,$$

permits construction of the solution for the first expansion coefficient in the stratum in the image space:

$$T^{(1)P} = \left\{K_1\,(z)\,\sqrt{p} + K_1\,(r)\,p\right\} T^{(0)P}\,, \tag{7.3}$$

where

$$K_1\,(z) = \frac{1}{2}\left[\frac{1}{6}\left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right) - \left(1 - \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right) z - \right.$$

$$\left. - \frac{1}{2}\left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right) z^2\right];$$

$$K_1\,(r) = \frac{\lambda_1 b\,(r^2 - r_0^2)}{8B}\left[\frac{1}{3}\left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)^2 + \left(1 - \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)^2\right]$$

($K_1(z)$ is a second-degree polynomial in the variable $z$, and $K_1(r)$ is a function of $r$).

The solution of problem (5.5)-(5.10) is substantially simplified on the averaging surface for $r = r_0$, when the function $K_1(r) = 0$. For the expansion coefficients in the stratum in the image space on the averaging surface, the following equation holds:

$$T^{(i)P} = K_i(z)\, p^{i/2}\, T^{(0)P}, \tag{7.4}$$

where $K_i(z)$ is a polynomial of degree $2i$ in the variable $z$. The first coefficient of the asymptotic expansion is obtained from Eqs. (5.5)-(5.10) for $r_0 = 0$.

Based on expression (7.3) it is established that the radius of convergence of the power series (2.1) in the image space is proportional to $1/\sqrt{p}$, which corresponds to $\sqrt{t}$ in the inverse transforms. Hence, the radius of convergence for long times reaches values much larger than unity. This implies that, for such times, the solutions also hold for ε larger than unity; therefore, in most practically important cases two terms are sufficient for obtaining solutions with a small error.

Converting to inverse transforms, from Eqs. (7.3), (7.1), and (7.2) we obtain for the first expansion

$$T^{(1)} = \frac{1}{\sqrt{\left(\pi\left(t - \frac{\lambda_1 b r^2}{2B}\right)\right)}} \left( K_1(z) + \frac{K_1(r)\,\lambda_1 r^2 \left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)}{8B\left(t - \frac{\lambda_1 b r^2}{2B}\right)} \right) \times$$

$$\times \exp\left( - \frac{\lambda_1^2 r^4 \left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)^2}{64B^2\left(t - \frac{\lambda_1 b r^2}{2B}\right)} \right) I\left(t - \frac{\lambda_1 b r^2}{2B}\right), \tag{7.5}$$

$$T_1^{(1)} = \frac{1}{\sqrt{\left(\pi\left(t - \frac{\lambda_1 b r^2}{2B}\right)\right)}} \times$$

$$\times \left( K_1(1) + \frac{K_1(r)\left[\frac{\lambda_1 r^2}{4B}\left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right) + (z-1)\right]}{4\left(t - \frac{\lambda_1 b r^2}{2B}\right)} \right) \times$$

$$\times \exp\left( - \frac{\left[\frac{\lambda_1 r^2}{4B}\left(1 + \sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right) + (z-1)\right]^2}{4\left(t - \frac{\lambda_1 b r^2}{2B}\right)} \right) I\left(t - \frac{\lambda_1 b r^2}{2B}\right), \tag{7.6}$$

$$T_2^{(1)} = \frac{1}{\sqrt{\left(\pi\left(t - \frac{\lambda_1 b r^2}{2B}\right)\right)}} \times$$

$$\times\left(K_1(-1)+\frac{K_1(r)\left[\frac{\lambda_1 r^2}{4B}\left(1+\sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)-\sqrt{\left(\frac{\lambda_1 b_2}{\lambda_2}\right)}(z+1)\right]}{4\left(t-\frac{\lambda_1 b r^2}{2B}\right)}\right)\times$$

$$\times\exp\left(-\frac{\left[\frac{\lambda_1 r^2}{4B}\left(1+\sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right)-\sqrt{\left(\frac{\lambda_1 b_2}{\lambda_2}\right)}(z+1)\right]^2}{4\left(t-\frac{\lambda_1 b r^2}{2B}\right)}\right)I\left(t-\frac{\lambda_1 b r^2}{2B}\right), \quad (7.7)$$

where

$$K_1(1)=-\frac{1}{3}\left(2-\sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}\right); \quad K_1(-1)=-\frac{1}{3}\left(2\sqrt{\left(\frac{\lambda_2 b_2}{\lambda_1}\right)}-1\right).$$

It is possible to verify the validity of expressions (7.5)-(7.7) by direct substitution into Eqs. (5.5)-(5.10).

The final solutions for the two approximations found are written as

$$T=T^{(0)}+\frac{\lambda_1}{\lambda}T^{(1)}, \quad T_1=T_1^{(0)}+\frac{\lambda_1}{\lambda}T_1^{(1)}, \quad T_2=T_2^{(0)}+\frac{\lambda_1}{\lambda}T_2^{(1)}. \quad (7.8)$$

The concentrated-capacity scheme, which was previously used for describing actual processes, contains certain approximations whose error cannot be assessed within the framework of the scheme itself. The method of a small parameter permits evaluation of the relative error of this scheme in the form

$$\delta=\frac{T-T^{(0)}}{T}=\frac{\varepsilon T^{(1)}}{T^{(0)}+\varepsilon T^{(1)}}; \quad \delta=\frac{T_j-T_j^{(0)}}{T_j}=\frac{\varepsilon T_j^{(1)}}{T_j^{(0)}+\varepsilon T_j^{(1)}}, \quad j=1,2. \quad (7.9)$$

It should be noted that higher-order corrections contribute much less to the error.

**8. Graphical Representation of Results.** Based on the analytical solutions obtained, graphs are constructed for space-time temperature distributions in the stratum and the surrounding rocks in the zeroth-order and first approximations. Graphs of the relative error of the zeroth-order approximation $\delta$ as a function of the coordinate $z$ are also presented.

In Fig. 2, the numeral 1 denotes curves corresponding to the zeroth-order approximation (4.17)-(4.19); 2, to the first coefficient of the expansion (7.5)-(7.7); 3, to the solution in the first approximation (7.8); and 4, to the relative error (7.9). All graphs are constructed in dimensionless coordinates for the following values of the physical quantities: volumetric heat capacity of the stratum, 695 kcal/($m^3\cdot{}^oC$), and of the surrounding rocks, 770 kcal/($m^3\cdot{}^oC$); thermal conductivity of the stratum, 2.23 kcal/(m·h·$^oC$), and of the surrounding rocks, 1 kcal/(m·h·$^oC$); water flow rate, 600 $m^3$/day; stratum thickness, 10 m [7]. All graphs are constructed for the distance $r$ = 20.

From curve 1 (Fig. 2a) for the zeroth-order approximation, $z$-independence of $T$ is observed in the stratum interval $-1 < z < 1$, which is what should occur in accordance with the concentrated-capacity scheme. The first expansion coefficient (curve 2) assumes both negative and positive values within the limits of the stratum. Owing to allowance for the correction, the solution in the first approximation (curve 3) reflects the temperature distribution in the stratum more realistically, which manifests itself in its dependence on $z$. It is seen from the figure that the zeroth-order approximation describes the temperature distribution in the stratum center in deficit and the temperature distribution along the stratum edges in excess. In the surrounding media, the zeroth-order approximation gives an excessive value of the temperature.

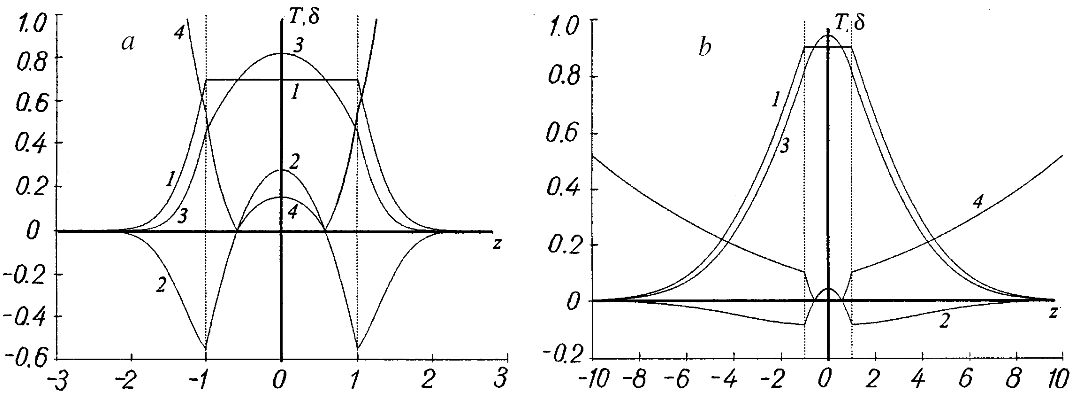

Fig. 2. Graphs of the temperature $T$ and the relative error $\delta$ as functions of the dimensionless coordinate $z$: a) $\varepsilon \approx 0.45$, $t$ = 0.5; b) $\varepsilon$ = 1, $t$ = 10.

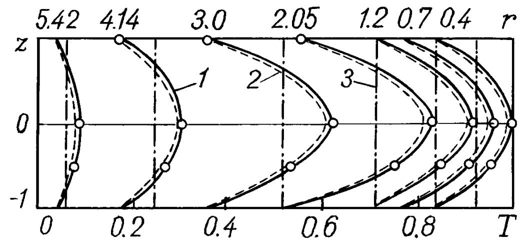

Fig. 3. Comparison of theoretical results with experimental data: 1) experimental curves; 2, 3) calculated curves in the first and zeroth-order approximation, respectively.

In the zeroth solution, there is a sharp change in the curve slope at the stratum boundaries. On the first-approximation curve 3, a slope discontinuity at the boundaries $z = \pm 1$ is visually detected, although this discontinuity is decreased in comparison with the zeroth-order approximation (curves 1 and 3).

Curve 4 reflects the dynamics of the relative errors for the stratum and the surrounding rocks. At two points of the stratum, where the curve of the first expansion coefficient intersects the $z$ axis, the error is zero. In the surrounding media the error increases, which is linked with the temperature decrease.

Figure 2b corresponds to fairly large values $\varepsilon$ = 1 ($\varepsilon < R$) at long times $t$ = 10. The curves of the zeroth-order (curve 1) and first (curve 3) approximations in the surrounding rocks are close, which implies that the method of a small parameter can be used with a small error even for large $\varepsilon$. The first approximation in the stratum rectifies the drawback of the concentrated-capacity scheme. Thus, with decrease in $\varepsilon$ and increase in the time, the error of the concentrated-capacity scheme decreases. For longer times, the discontinuity of the curves at the stratum boundaries in the first approximation is reduced, and conditions (1.4) are fulfilled more accurately.

**9. Comparison of Results of Calculating the Temperature Fields Based on the Method of a Small Parameter with Experimental Data.** Experimental data for the temperature fields in oil strata are known in the literature [1, 8]. We took these data as a basis for comparison of theory and experiment.

Figure 3 presents measurement results (continuous curves) and results calculated for the temperature by the method of a small parameter in the first approximation on the calculation surface at $r = r_0$ (dashed curves). The calculation was carried out by Eq. (7.8) with account for Eqs. (4.17) and (7.5). The dimensionless temperature is laid off as the abscissas, and the distance from the stratum bottom is laid off as the ordinates; the markers on the curves correspond to the distance from the pressure well expressed in units of the stratum thickness. These curves correspond to the dimensionless time $t$ = 0.245.

Theoretical curves are constructed for the same instant of time and the same distances from the axis of the pressure well. It is seen from the figure that the discrepancy between the experimental and theoretical curves is insignificant and is no greater than 10%. Such good agreement between the theoretical and experimental curves indicates that the application of the method of a small parameter provided adequate calculating equations even in the first approximation.

The same figure presents calculated curves (dashed lines) in the zeroth-order approximation by Eq. (4.17). As follows from the theory, these curves correspond to the mean values of the stratum temperature.

The results of the comparison of the theoretical results and the experimental data indicate that the method of a small parameter can be used for calculating the temperatuire fields in deep-lying strata.

## CONCLUSIONS

1. The method of a small parameter can be used with success for solving a class of heat-conduction problems describing the temperature fields in oil and gas strata.

2. It is established that use of the method of a small parameter requires additional conditions for the problem. We managed to obtain higher-order approximations using a new integral condition that follows from coincidence of the mean temperatures in the stratum cross section and the zeroth-order approximation.

3. It is shown that, in the Laplace–Carson image space, the coefficients of expansion of the solution in $\varepsilon$ of order $i$ contain products of the zeroth-order expansion coefficient, a polynomial in the variable $z$ of degree $2i$, and the parameter of the Laplace–Carson transform $p^{i/2}$. The time dependence of the radius of convergence is found.

4. Based on the analytical solutions obtained, the error of the concentrated-capacity scheme is evaluated for the first time, and the ranges of its application to practical calculations are determined. With the aid of the method of a small parameter, it is proved that the scheme accuracy increases with increase in the time and with decrease in the ratio of the thermal conductivities of the surrounding rocks and the stratum $\lambda_1/\lambda$. It is demonstrated that, for relative times $t >> 1$, it is sufficient to retain two terms in the resultant solution. Analytical expressions are found for the coefficients of expansion of the sought solutions of the problem for the zeroth and first orders.

## NOTATION

$z_d$, $r_d$ ($z$, $r$), dimensional (dimensionless) cylindrical coordinates; $\tau$ ($t$), dimensional (dimensionless) time; $2h$, stratum thickness; $T_d$, $T_{d1}$, and $T_{d2}$ ($T$, $T_1$, and $T_2$), temperatures (dimensionless temperatures) in the stratum and the surrounding rocks, respectively; $T_0$, temperature of the injected fluid; $\lambda$, $\lambda_1$, and $\lambda_2$, thermal conductivities in the stratum and the surrounding rocks, respectively; $c$, $c_1$, and $c_2$, specific heats of the stratum and the surrounding rocks; $\rho$, $\rho_1$, and $\rho_2$, densities of the stratum and the surrounding rocks; $Q$, fluid flow rate; $p$, parameter of the Laplace–Carson transform; $\text{erfc}\,(x) = \frac{2}{\sqrt{\pi}}\, c \int_x^\infty \exp(-u^2)du$; $I(t)$, Heaviside unit function. Subscripts: 1 and 2, overlying and underlying rocks.

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